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Coherent State and Inhomogeneous Differential Realization of the SPL(2,1) Superalgebra

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The simple coherent state of the SPL(2,1) superalgebra is constructed and its properties are discussed in detail. The matrix elements of the SPL(2,1) generators in the coherent state space are calculated. A new form for the homogeneous differential realization of the SPL(2,1) superalgebra is given.

KEY WORDS: coherent state; differential realization SPL (2, 1) superalgebra.

1. INTRODUCTION

Recently, much attention has been paid to the coherent states of Lie (super) algebra (Balantekin and Bars, 1982; Balantekin *et al*., 1989; Blantekin *et al*., 1988; Fatyga *et al*., 1991; Gilmore *et al*., 1975; Glauber, 1963; Klauder and Skagerstam, 1985; Perelomov, 1972, 1975; Quesne, 1986, 1990; Radicliffe, 1971). Recently discovered quasi-exactly solvable problems (QESP) in quantum mechanics have become increasingly important because they have been generalized to study the conformal field theory (Morozov *et al*., 1990). A connection of QESP and finite-dimensional inhomogeneous differential realizations of Lie algebras (or superalgebras) has been described at the first time by Turbiner (Shifman and Turbiner, 1989; Turbiner, 1988, 1992; Turbiner and Ushveridze, 1987). Turbner gave a complete classification of the one-dimensional QESP by making use of the inhomogeneous differential realization of the SU(2) algebra, and pointed out that the multidimensional QESP may be studied and the general procedure to construct the multidimensional QESP in terms of the inhomogeneous differential realizations of the Lie superalgebra was presented (Brihaye *et al*., 1997; Shifman and Turbiner, 1989; Turbiner, 1988, 1992; Turbiner and Ushveridze, 1987). The key of the solution of the QESP lies in studying finite-dimensional inhomogeneous differential realizations of Lie (super)algebras. Therefore, it is very important to study the

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inhomogeneous differential realizations of Lie superalgebras. A supercoherent state and inhomogeneous differential realizations have been given in Chen (1993, 2000, 2001). The purpose of the present paper is to derive further the new inhomogeneous differential realization of the SPL(2,1) superalgebra on the basis of the simple coherent states. In the present paper we shall first construct the simple coherent states of the SPL(2,1) superalgebra and discuss their properties. Then we calculate the matrix elements of the SPL(2,1) generators in the coherent state representation and give a new form of the inhomogeneous differential realization of the SPL(2,1) in the coherent-state spaces.

2. THE SPL(2,1) COHERENT STATES AND PROPERTIES

In accordance with Fu and Sun (1990, 1991) the generators of the SPL(2,1) superalgebra read as follows:

$$\{Q_3,\, Q_+,\, Q_-,\, B \in \text{SPL}(2,1)_{\bar{0}} \mid V_+,\, V_-,\, W_+,\, W_- \in \text{SPL}(2,1)_{\bar{1}}\} \tag{2.1}$$

and satisfy the following commutation and anticommutation relations:

$$\begin{aligned}
&[Q_3,\, Q_\pm] = \pm Q_\pm, \quad [Q_+,\, Q_-] = 2Q_3, \quad [B,\, Q_\pm] = [B,\, Q_3] = 0,\\
&[Q_3,\, V_\pm] = \pm\frac{1}{2}V\pm, \quad [Q_3,\, W_\pm] = \pm\frac{1}{2}W_\pm, \quad [B,\, V_\pm] = \frac{1}{2}V_\pm,\\
&[B,\, W_\pm] = -\frac{1}{2}W_\pm, \quad [Q_\pm,\, V_\mp] = V_\pm, \quad [Q_\pm,\, W_\mp] = W_\pm, \quad [Q_\pm,\, V_\pm] = 0,\\
&[Q_\pm,\, W_\pm] = 0, \quad \{V_\pm,\, V_\pm\} = \{V_\pm,\, V_\mp\} = \{W_\pm,\, W_\pm\} = \{W_\pm,\, W_\mp\} = 0,\\
&\{V_\pm,\, W_\pm\} = \pm Q_\pm, \qquad \{V_\pm,\, W_\mp\} = -Q_3 \pm B.
\end{aligned} \tag{2.2}$$

According to Chen (1993) and relabelling the basis vector $\phi(k, \alpha_1, \alpha_2)$ of the finitedimensional irreducible representation of the SPL(2,1) superalgebra by $|\mathtt{N}, \mathtt{k}, \alpha_1, \alpha_2\rangle$ the actions of the generators on the basis vectors are

$$\begin{aligned}
Q_3|\mathtt{N}, \mathtt{k}, \alpha_1, \alpha_2\rangle &= \left(-\frac{N}{2} + k + \frac{\alpha_1}{2} + \frac{\alpha_2}{2}\right)|\mathtt{N}, \mathtt{k}, \alpha_1, \alpha_2\rangle\\
B|\mathtt{N}, \mathtt{k}, \alpha_1, \alpha_2\rangle &= \frac{1}{2}(\alpha_2 - \alpha_1)|\mathtt{N}, \mathtt{k}, \alpha_1, \alpha_2\rangle\\
Q_+|\mathtt{N}, \mathtt{k}, \alpha_1, \alpha_2\rangle &= (N - k - \alpha_1 - \alpha_2)|\mathtt{N}, \mathtt{k} + 1, \alpha_1, \alpha_2\rangle\\
Q_-|\mathtt{N}, \mathtt{k}, \alpha_1, \alpha_2\rangle &= k|\mathtt{N}, \mathtt{k} - 1, \alpha_1, \alpha_2\rangle\\
V_+|\mathtt{N}, \mathtt{k}, \alpha_1, \alpha_2\rangle &= \frac{1}{\sqrt{2}}(-1)^{\alpha_1}(N - k - \alpha_1)(1 - \alpha_2)|\mathtt{N}, \mathtt{k}, \alpha_1, \alpha_2 + 1\rangle\\
&\quad + \frac{1}{\sqrt{2}}\alpha_1|\mathtt{N}, \mathtt{k} + 1, \alpha_1 - 1, \alpha_2\rangle
\end{aligned}$$

$$
\begin{aligned}
V_-|\mathtt{N}, \mathtt{k}, \alpha_1, \alpha_2\rangle &= \frac{1}{\sqrt{2}}\alpha_1|\mathtt{N}, \mathtt{k}, \alpha_1 - 1, \alpha_2\rangle \\
&\quad - \frac{1}{\sqrt{2}}(-1)^{\alpha_1}(1-\alpha_2)\,k|\mathtt{N}, \mathtt{k}-1, \alpha_1, \alpha_2+1\rangle \\
W_+|\mathtt{N}, \mathtt{k}, \alpha_1, \alpha_2\rangle &= \frac{1}{\sqrt{2}}(N-k-\alpha_2)(1-\alpha_1)|\mathtt{N}, \mathtt{k}, \alpha_1+1, \alpha_2\rangle \\
&\quad + \frac{1}{\sqrt{2}}(-1)^{\alpha_1}\alpha_2|\mathtt{N}, \mathtt{k}+1, \alpha_1, \alpha_2-1\rangle \\
W_-|\mathtt{N}, \mathtt{k}, \alpha_1, \alpha_2\rangle &= \frac{1}{\sqrt{2}}(-1)^{\alpha_1}\alpha_2|\mathtt{N}, \mathtt{k}, \alpha_1, \alpha_2-1\rangle \\
&\quad - \frac{1}{\sqrt{2}}(1-\alpha_1)k|\mathtt{N}, \mathtt{k}-1, \alpha_1+1, \alpha_2\rangle
\end{aligned} \tag{2.3}
$$

where

$$
\{|\mathtt{N}, \mathtt{k}, \alpha_1, \alpha_2\rangle | \mathtt{k}^+\alpha_1^+\alpha_2 \leq \mathtt{N}, \quad \mathtt{N} \in Z^+, \quad k = 0, 1, 2, \ldots, \quad \alpha_1, \alpha_2 = 0, 1\}
$$

and

$$
k = \begin{cases} 0, 1, \ldots, N & \text{when } \alpha_1 = 0, \alpha_2 = 0 \\ 0, 1, \ldots, N-1 & \text{when } \alpha_1 = 0, \alpha_2 = 1 \\ 0, 1, \ldots, N-1 & \text{when } \alpha_1 = 1, \alpha_2 = 0 \\ 0, 1, \ldots, N-2 & \text{when } \alpha_1 = 1, \alpha_2 = 1 \end{cases} \tag{2.4}
$$

The space $\{|\mathtt{N}, \mathtt{k}, \alpha_1, \alpha_2\rangle\}$ of the irrep N of the SPL(2,1) superalgebra is $4N$ dimensional and may be divided into four subspaces $\{|\mathtt{N}, \mathtt{k}, 0, 0\rangle\}$, $\{|\mathtt{N}, \mathtt{k}, 0, 1\rangle\}$, $\{|\mathtt{N}, \mathtt{k}, 1, 0\rangle\}$, and $\{|\mathtt{N}, \mathtt{k}, 1, 1\rangle\}$ corresponding to $(\alpha_1, \alpha_2) = (0, 0), (0, 1), (1, 0)$, and (1,1), respectively. All the basis vectors $|\mathtt{N}, \mathtt{k}, \alpha_1, \alpha_2\rangle$ are assumed to be normalized as

$$
\binom{N-\alpha_1-\alpha_2}{k} \langle N, k, \alpha_1, \alpha_2 \mid N, k, \alpha_1, \alpha_2\rangle = 1 \tag{2.5}
$$

The completeness condition of the vectors of the irrep may be expressed as

$$
\sum_{(\alpha_1,\alpha_2),k=0}^{N-\alpha_1-\alpha_2} \binom{N-\alpha_1-\alpha_2}{k} |N, k, \alpha_1, \alpha_2\rangle\langle N, k, \alpha_1, \alpha_2| = I \tag{2.6}
$$

where I is the identity operator.

One can easily show the following formulas from Eq. (2.3)

$$
Q_+^n|N, 0, \alpha_1, \alpha_2\rangle = \binom{N-\alpha_1-\alpha_2}{n} n!|N, n, \alpha_1, \alpha_2\rangle \tag{2.7}
$$

where

$$\binom{N}{n} = \frac{N!}{(N-n)!n!}$$

In terms of Bloch's method we now define the coherent state of the SPL(2,1) by applying the exponential operator $\exp(ZQ_+)$ on the lowest-weight state $|N, 0, \alpha_1, \alpha_2\rangle$ of its irrep

$$|\mathrm{Z}, \alpha_1, \alpha_2\rangle = \mathrm{C}(\mathrm{Z}, \alpha_1, \alpha_2)\ \exp(ZQ_+)|N, 0, \alpha_1, \alpha_2\rangle \tag{2.8}$$

where $\mathrm{C}(\mathrm{Z}, \alpha_1, \alpha_2)$ is a normalization constant to be determined.

Using the formulas Eq. (2.7), the coherent state (2.8) may be rewritten as follows:

$$|\mathrm{Z}, \alpha_1, \alpha_2\rangle = \mathrm{C}(\mathrm{Z}, \alpha_1, \alpha_2) \sum_{n=0}^{N-\alpha_1-\alpha_2} \binom{N-\alpha_1-\alpha_2}{n} Z^n |N, n, \alpha_1, \alpha_2\rangle \tag{2.9}$$

We require that the SPL(2,1) coherent state defined in this way are normalized in the form

$$\langle Z, \alpha_1, \alpha_2 \mid Z, \alpha_1, \alpha_2\rangle = 1 \tag{2.10}$$

It follows from Eqs. (2.9) and (2.10) that

$$\mathrm{C}(\mathrm{z}, \alpha_1, \alpha_2) = (1 + \bar{\mathrm{z}}\mathrm{z})^{-\frac{1}{2}(N-\alpha_1-\alpha_2)} \tag{2.11}$$

The scalar product of two coherent states is of the form

$$\langle z', \alpha_1', \alpha_2' \mid z, \alpha_1, \alpha_2\rangle = C(z', \alpha_1', \alpha_2')\, \mathrm{C}(\mathrm{z}, \alpha_1, \alpha_2)(1 + \bar{z}'z)^{N-\alpha_1-\alpha_2} \delta_{\alpha_1', \alpha_1} \delta_{\alpha_2', \alpha_2} \tag{2.12}$$

The expansion coefficients of the coherent state $|\mathrm{Z}, \alpha_1, \alpha_2\rangle$ may be found in terms of the complete orthonormal set $\{|\mathrm{N}, \mathrm{k}, \alpha_1, \alpha_2\rangle\}$. Thus, we have

$$\langle N, k, \alpha_1, \alpha_2 \mid z, \alpha_1, \alpha_2\rangle = \mathrm{C}(\mathrm{z}, \alpha_1, \alpha_2)\mathrm{z}^k \tag{2.13}$$

While orthogonality is a convenient property for a set of basis vectors it is not a necessary one. The essential property of such a set is that it be complete. Since the $4N$ state vectors $\{|\mathrm{N}, \mathrm{k}, \alpha_1, \alpha_2\rangle\}$ of an irrep of the SPL(2,1) superalgebra are known to form a completeness orthogonal set, the set of the coherent states $\{|\mathrm{Z}, \alpha_1, \alpha_2\rangle\}$ for the SPL(2,1) superalgebra can be shown without difficulty to form a complete set. To give a proof we need only demonstrate that the unit operator may be expressed as a suitable sum or an integral, over the complex Z plane, of projection operators of the form $|\mathrm{Z}, \alpha_1, \alpha_2\rangle\langle Z, \alpha_1, \alpha_2|$. In order to describe such integral we introduce generally the differential element of weight area in the Z

plane

$$d^2\sigma(Z,\alpha_1,\alpha_2) = \sigma(|Z|,\alpha_1,\alpha_2)d^2(Z,\alpha_1,\alpha_2) = \sigma(|Z|,\alpha_1,\alpha_2)d(\mathrm{Re}\,Z)d(\mathrm{Im}\,Z) \tag{2.14}$$

If we set $Z = |Z|\,e^{i\theta}$, then we may rewrite Eq. (2.14) as

$$d^2\sigma(Z,\alpha_1,\alpha_2) = \sigma(|Z|,\alpha_1,\alpha_2)|Z|d|Z|d\theta \tag{2.15}$$

The problem here may by changed to find the weight function $\sigma(Z,\alpha_1,\alpha_2)$ such that

$$\int d^2\sigma(Z,\alpha_1,\alpha_2)|Z,\alpha_1,\alpha_2\rangle\langle Z,\alpha_1,\alpha_2| = I \tag{2.16}$$

Let |f⟩ and |g⟩ be two arbitrary vectors, then Eq. (2.16) means that

$$\langle \mathrm{f} \mid \mathrm{g}\rangle = \int d^2\sigma(Z,\alpha_1,\alpha_2)\langle f \mid Z,\alpha_1,\alpha_2\rangle\langle Z,\alpha_1,\alpha_2 \mid \mathrm{g}\rangle \tag{2.17}$$

Substituting the definition Eq. (2.9) into Eq. (2.17) and integrating over the entire area of the complex plane we have

$$\begin{aligned}\langle \mathrm{f} \mid \mathrm{g}\rangle = 2\pi \sum_{n=0,\alpha_1,\alpha_2}^{N-\alpha_1-\alpha_2} &\binom{N-\alpha_1-\alpha_2}{n}\binom{N-\alpha_1-\alpha_2}{n} \\ &\times \int_0^\infty |Z|^{2n+1}\sigma(|Z|,\alpha_1,\alpha_2)(1+|Z|^2)^{-N+\alpha_1+\alpha_2}d|Z| \\ &\times \langle \mathrm{f} \mid N,n,\alpha_1,\alpha_2\rangle\langle N,n,\alpha_1,\alpha_2 \mid \mathrm{g}\rangle\end{aligned} \tag{2.18}$$

Comparing Eq. (2.18) with Eq. (2.6) we must have

$$2\pi\binom{N-\alpha_1-\alpha_2}{n}\int_0^\infty |\mathrm{z}|^{2n+1}(1+|\mathrm{z}|^2)^{-N+\alpha_1+\alpha_2}\sigma(|\mathrm{z}|,\alpha_1,\alpha_2)d|\mathrm{z}| = 1 \tag{2.19}$$

With the aid of the following integral identity,

$$\int_0^\infty \frac{x^{2n+1}}{(1+x^x)^m}dx = \frac{n!(m-n-2)!}{2(m-1)!} \tag{2.20}$$

and by comparing Eq. (2.19) with Eq. (2.20) we finally obtain the weight function

$$\sigma(|Z|,\alpha_1,\alpha_2) = \frac{N-\alpha_1-\alpha_2+1}{\pi(1+|Z|^2)^2} \tag{2.21}$$

We have thus shown

$$\frac{1}{x}\int d^2(\mathrm{z},\alpha_1,\alpha_2)\frac{N-\alpha_1-\alpha_2+1}{(1+|\mathrm{z}|^2)^2}|\mathrm{z},\alpha_1,\alpha_2\rangle\langle \mathrm{z},\alpha_1,\alpha_2| = 1 \tag{2.22}$$

which is a completeness relation for the coherent states of the SPL(2,1) superalgebra of precisely the type desired. As a result of the above completeness relation, an arbitrary vector $|\Psi\rangle$ can be expanded in terms of the coherent states for the SPL(2,1) superalgebra. To secure the expansion of $|\Psi\rangle$ in terms of the coherent states $\{|Z, \alpha_1, \alpha_2\rangle\}$, we multiply $|\Psi\rangle$ by the expression (2.22) of the unit operator. We then find

$$|\Psi\rangle = \frac{1}{\pi}\int d^2(z,\alpha_1,\alpha_2)\frac{N-\alpha_1-\alpha_2+1}{(1+|z|^2)^2}|z,\alpha_1,\alpha_2\rangle\langle z,\alpha_1,\alpha_2 \mid \Psi\rangle \tag{2.23}$$

3. MATRIX ELEMENTS OF THE GENERATORS

The present section will be devoted to calculating the matrix elements of the SPL(2,1) generators in the coherent-state representation. The calculation results are as follows:

$$\begin{aligned}
\langle z',\alpha_1',\alpha_2'|Q_3|z,\alpha_1,\alpha_2\rangle &= -\frac{1}{2}C(z',\alpha_1',\alpha_2')\,C(z,\alpha_1,\alpha_2)(N-\alpha_1-\alpha_2)\\
&\quad\times(1-\overline{z'}z)(1+\overline{z'}z)^{N-\alpha_1-\alpha_2-1}\delta_{\alpha_1',\alpha_1}\delta_{\alpha_2',\alpha_2}\\
\langle z',\alpha_1',\alpha_2'|Q_+|z,\alpha_1,\alpha_2\rangle &= C(z',\alpha_1',\alpha_2')\,C(z,\alpha_1,\alpha_2)(N-\alpha_1-\alpha_2)\overline{z'}\\
&\quad\times(1+\overline{z'}z)^{N-\alpha_1-\alpha_2-1}\delta_{\alpha_1',\alpha_1}\delta_{\alpha_2',\alpha_2}\\
\langle z',\alpha_1',\alpha_2'|Q_-|z,\alpha_1,\alpha_2\rangle &= C(z',\alpha_1',\alpha_2')\,C(z,\alpha_1,\alpha_2)(N-\alpha_1-\alpha_2)z\\
&\quad\times(1+\overline{z'}z)^{N-\alpha_1-\alpha_2-1}\delta_{\alpha_1',\alpha_1}\delta_{\alpha_2',\alpha_2}\\
\langle z',\alpha_1',\alpha_2'|B|z,\alpha_1,\alpha_2\rangle &= \frac{1}{2}C(z',\alpha_1',\alpha_2')\,C(z,\alpha_1,\alpha_2)(\alpha_2-\alpha_1)(1+\overline{z'}z)^{N-1}\\
&\quad\times\delta_{\alpha_1',\alpha_1}\delta_{\alpha_2',\alpha_2}\\
\langle z',\alpha_1',\alpha_2'|V_+|z,\alpha_1,\alpha_2\rangle &= \frac{1}{\sqrt{2}}(-1)^{\alpha_1}(N-\alpha_1)\,C(z',\alpha_1',\alpha_2')\,C(z,\alpha_1,\alpha_2)\\
&\quad\times(1+\overline{z'}z)^{N-\alpha_1-1}\delta_{\alpha_1',\alpha_1}\delta_{\alpha_2',\alpha_2+1}\\
&\quad+\frac{1}{\sqrt{2}}C(z',\alpha_1',\alpha_2')\,C(z,\alpha_1,\alpha_2)\overline{z'}(1+\overline{z'}z)^{N-\alpha_2-1}\\
&\quad\times\delta_{\alpha_1',\alpha_1-1}\delta_{\alpha_2',\alpha_2}\\
\langle z',\alpha_1',\alpha_2'|V_-|z,\alpha_1,\alpha_2\rangle &= \frac{1}{\sqrt{2}}C(z',\alpha_1',\alpha_2')\,C(z,\alpha_1,\alpha_2)(1+\overline{z'}z)^{N-\alpha_2-1}\\
&\quad\times\delta_{\alpha_1',\alpha_1-1}\delta_{\alpha_2',\alpha_2}
\end{aligned}$$

$$
\begin{aligned}
&\quad - \frac{1}{\sqrt{2}}(-1)^{\alpha_1}(N-\alpha_1)\, C(z', \alpha_1', \alpha_2')\, \mathrm{C}(\mathrm{z}, \alpha_1, \alpha_2)\mathrm{z} \\
&\quad \times (1+\overline{z'}z)^{N-\alpha_1-1}\delta_{\alpha_1', \alpha_1}\delta_{\alpha_2', \alpha_2+1} \\
\langle z', \alpha_1', \alpha_2'|W_+|z, \alpha_1, \alpha_2\rangle &= \frac{1}{\sqrt{2}}(N-\alpha_2)\, C(z', \alpha_1', \alpha_2')\, \mathrm{C}(\mathrm{z}, \alpha_1, \alpha_2) \\
&\quad \times (1+\overline{z'}z)^{N-\alpha_2-1}\delta_{\alpha_1'+1, \alpha_1}\delta_{\alpha_2', \alpha_2} \\
&\quad + \frac{1}{\sqrt{2}}(-1)^{\alpha_1} C(z', \alpha_1', \alpha_2')\, \mathrm{C}(\mathrm{z}, \alpha_1, \alpha_2)\overline{z'} \\
&\quad \times (1+\overline{z'}z)^{N-\alpha_1-1}\delta_{\alpha_1', \alpha_1}\delta_{\alpha_2', \alpha_2-1} \\
\langle z', \alpha_1', \alpha_2'|W_-|z, \alpha_1, \alpha_2\rangle &= \frac{1}{\sqrt{2}}(-1)^{\alpha_1} C(z', \alpha_1', \alpha_2')\, \mathrm{C}(\mathrm{z}, \alpha_1, \alpha_2) \\
&\quad \times (1+\overline{z'}z)^{N-\alpha_1-1}\delta_{\alpha_1', \alpha_1}\delta_{\alpha_2', \alpha_2-1} \\
&\quad - \frac{1}{\sqrt{2}}(N-\alpha_2)\, C(z', \alpha_1', \alpha_2') \\
&\quad \times \mathrm{C}(\mathrm{z}, \alpha_1, \alpha_2)\mathrm{z}(1+\overline{z'}z)^{N-\alpha_2-1}\delta_{\alpha_1', \alpha_1+1}\delta_{\alpha_2', \alpha_2}
\end{aligned} \tag{3.1}
$$

In evaluating the matrix elements, one needs only use Eqs. (2.3), (2.5), and (2.9), for example,

$$
\begin{aligned}
\langle z', \alpha_1', \alpha_2'|Q_3|z, \alpha_1, \alpha_2\rangle &= C\left(z', \alpha_1', \alpha_2'\right) \mathrm{C}(\mathrm{z}, \alpha_1, \alpha_2) \\
&\quad \times \sum_{m=0}^{N-\alpha_1'-\alpha_2'} \sum_{n=0}^{N-\alpha_1-\alpha_2} \binom{N-\alpha_1'-\alpha_2'}{m} \binom{N-\alpha_1-\alpha_2}{n} \\
&\quad \times (\overline{z'})^m z^n \langle N, m, \alpha_1', \alpha_2'|Q_3|N, n, \alpha_1, \alpha_2\rangle \\
&= C(z', \alpha_1', \alpha_2')\, \mathrm{C}(\mathrm{z}, \alpha_1, \alpha_2) \sum_{n=0}^{N-\alpha_1-\alpha_2} \binom{N-\alpha_1-\alpha_2}{n} \\
&\quad \times \left\{ -\frac{1}{2}(N-\alpha_1-\alpha_2) + n \right\} (\overline{z'}, z)^n \\
&= C\left(z', \alpha_1', \alpha_2'\right) \mathrm{C}(\mathrm{z}, \alpha_1, \alpha_2) \left\{ \frac{1}{2}(N-\alpha_1-\alpha_2) \right. \\
&\quad \times \sum_{n=0}^{N-\alpha_1-\alpha_2} \binom{N-\alpha_1-\alpha_2}{n} (\overline{z'}, z)^n - (N-\alpha_1-\alpha_2)
\end{aligned}
$$

$$\times \sum_{n=0}^{N-\alpha_1-\alpha_2-1} \binom{N-\alpha_1-\alpha_2-1}{n} (\bar{z}', z)^n \Bigg\}$$

$$= -\frac{1}{2}(N-\alpha_1-\alpha_2)\, C\left(z', \alpha_1', \alpha_2'\right) \mathrm{C}(\mathrm{z}, \alpha_1, \alpha_2)$$

$$\times (1-\bar{z}'z)(1+\bar{z}'z)^{N-\alpha_1-\alpha_2-1} \delta_{\alpha_1',\alpha_1} \delta_{\alpha_2',\alpha_2} \tag{3.2}$$

4. THE INHOMOGENEOUS DIFFERENTIAL REALIZATION OF THE SPL(2,1)

We now consider the actions of the SPL(2,1) generators on the coherent state $\{|\mathrm{Z}, \alpha_1, \alpha_2\rangle\}$. By making use of Eq. (2.3) and the completeness relation, Eq. (2.6), and the following recurrence formulas

$$(N-n)\binom{N}{n} = N\binom{N-1}{n}(N-n)\binom{N}{n}$$

$$= (n+1)\binom{N}{n+1} n\binom{N}{n} = (N-n+1)\binom{N}{n-1} \tag{4.1}$$

we easily obtain the following results

$$Q_3|z, \alpha_1, \alpha_2\rangle = \left\{ -\frac{1}{2}(N-\alpha_1-\alpha_2)\frac{1}{1+\bar{z}z} + z\frac{d}{dz} \right\} |z, \alpha_1, \alpha_2\rangle$$

$$B|z, \alpha_1, \alpha_2\rangle = \frac{1}{2}(\alpha_2-\alpha_1)|z, \alpha_1, \alpha_2\rangle$$

$$Q_+|z, \alpha_1, \alpha_2\rangle = \left\{ \frac{1}{2}(N-\alpha_1-\alpha_2)\frac{\bar{z}}{1+\bar{z}z} + \frac{d}{dz} \right\} |z, \alpha_1, \alpha_2\rangle$$

$$Q_-|z, \alpha_1, \alpha_2\rangle = \left\{ \frac{1}{2}(N-\alpha_1-\alpha_2)\, z\frac{2+\bar{z}z}{1+\bar{z}z} - z^2\frac{d}{dz} \right\} |z, \alpha_1, \alpha_2\rangle$$

$$V_+|z, \alpha_1, \alpha_2\rangle = \frac{1}{\sqrt{2}}(-1)^{\alpha_1}(N-\alpha_1)\frac{1}{(1+\bar{z}z)^{1/2}}|z, \alpha_1, \alpha_2+1\rangle$$

$$+ \frac{1}{\sqrt{2}}\frac{1}{(1+\bar{z}z)^{1/2}} \left\{ \frac{1}{2}\bar{z} + \frac{1}{N-\alpha_2}(1+\bar{z}z)\frac{d}{dz} \right\} |z, \alpha_1-1, \alpha_2\rangle$$

$$V_-|z, \alpha_1, \alpha_2\rangle = -\frac{1}{\sqrt{2}}(-1)^{\alpha_1}(N-\alpha_1)\, z\frac{1}{(1+\bar{z}z)^{1/2}}|z, \alpha_1, \alpha_2+1\rangle$$

$$+ \frac{1}{\sqrt{2}}\frac{1}{(1+\bar{z}z)^{1/2}}$$

$$\times\left\{1+\frac{1}{2}\bar{z}z+\frac{1}{N-\alpha_2}(1+\bar{z}z)\,z\frac{d}{dz}\right\}|z,\alpha_1-1,\alpha_2\rangle$$

$$W_+|z,\alpha_1,\alpha_2\rangle=\frac{1}{\sqrt{2}}(N-\alpha_2)\frac{1}{(1+\bar{z}z)^{1/2}}|z,\alpha_1+1,\alpha_2\rangle$$
$$+\frac{1}{\sqrt{2}}(-1)^{\alpha_1}\frac{1}{(1+\bar{z}z)^{1/2}}$$
$$\times\left\{\frac{1}{2}\bar{z}+\frac{1}{N-\alpha_1}(1+\bar{z}z)\frac{d}{dz}\right\}|z,\alpha_1,\alpha_2-1\rangle$$

$$W_-|z,\alpha_1,\alpha_2\rangle=-\frac{1}{\sqrt{2}}(N-\alpha_2)z\frac{1}{(1+\bar{z}z)^{1/2}}|z,\alpha_1+1,\alpha_2\rangle$$
$$+\frac{1}{\sqrt{2}}(-1)^{\alpha_1}\frac{1}{(1+\bar{z}z)^{1/2}}$$
$$\times\left\{1+\frac{1}{2}\bar{z}z-\frac{1}{N-\alpha_1}(1+\bar{z}z)z\frac{d}{dz}\right\}|z,\alpha_1,\alpha_2-1\rangle \qquad (4.2)$$

It may be worth noting at this point that many of the foregoing formulas can be abbreviated somewhat by adopting a normalization different from the conventional one for the coherent states. If we introduce the symbol $\|z,\alpha_1,\alpha_2\rangle$ for the states normalized in the new way and define these as

$$\|z,\alpha_1,\alpha_2\rangle=\sum_{n=0}^{N-\alpha_1-\alpha_2}\binom{N-\alpha_1-\alpha_2}{n}Z^n|N,n,\alpha_1,\alpha_2\rangle \qquad (4.3)$$

then we may write the scalar product of two such states as $\langle z',\alpha_1',\alpha_2'\|z,\alpha_1,\alpha_2\rangle$. We see from Eq. (4.3) that these scalar products are

$$\langle z',\alpha_1',\alpha_2'\|z,\alpha_1,\alpha_2\rangle=(1+\bar{z}'z)^{N-\alpha_1-\alpha_2}\delta_{\alpha_1',\alpha_1}\delta_{\alpha_2',\alpha_2} \qquad (4.4)$$

In accordance with the aforesaid consideration, it is clear that granted that we make C(z, α_1, α_2) $=1$ in the Eq. (2.9) that we can obtain a simple inhomogeneous differential realization of the SPL(2,1) superalgebra in the new coherent state space $\{\|z,\alpha_1,\alpha_2\rangle\}$. We now consider the actions of the SPL(2,1) generators the results are as follows:

$$Q_3\|z,\alpha_1,\alpha_2\rangle=\left\{-\frac{1}{2}(N-\alpha_1-\alpha_2)+z\frac{d}{dz}\right\}\|z,\alpha_1,\alpha_2\rangle$$

$$B\|z,\alpha_1,\alpha_2\rangle=\frac{1}{2}(\alpha_2-\alpha_1)\|z,\alpha_1,\alpha_2\rangle$$

$$Q_+\|z,\alpha_1,\alpha_2\rangle=\frac{d}{dz}\|z,\alpha_1,\alpha_2\rangle$$

$$
\begin{aligned}
Q_-\|z,\alpha_1,\alpha_2\rangle &= \left\{(N-\alpha_1-\alpha_2)\,z - z^2\frac{d}{dz}\right\}\|z,\alpha_1,\alpha_2\rangle \\
V_+\|z,\alpha_1,\alpha_2\rangle &= \frac{1}{\sqrt{2}}(-1)^{\alpha_1}(1-\alpha_2)(N-\alpha_1)\|z,\alpha_1,\alpha_2+1\rangle \\
&\quad + \frac{1}{\sqrt{2}}\,\alpha_1\frac{1}{N-\alpha_2}\frac{d}{dz}\|z,\alpha_1-1,\alpha_2\rangle \\
V_-\|z,\alpha_1,\alpha_2\rangle &= \frac{1}{\sqrt{2}}\alpha_1\left(1-\frac{1}{N-\alpha_2}z\frac{d}{dz}\right)\|z,\alpha_1-1,\alpha_2\rangle \\
&\quad - \frac{1}{\sqrt{2}}(-1)^{\alpha_1}(1-\alpha_2)(N-\alpha_1)z\|z,\alpha_1,\alpha_2+1\rangle \\
W_+\|z,\alpha_1,\alpha_2\rangle &= \frac{1}{\sqrt{2}}(1-\alpha_1)(N-\alpha_2)\|z,\alpha_1+1,\alpha_2\rangle \\
&\quad + \frac{1}{\sqrt{2}}(-1)^{\alpha_1}\alpha_2\frac{1}{N-\alpha_1}\frac{d}{dz}\|z,\alpha_1,\alpha_2-1\rangle \\
W_-\|z,\alpha_1,\alpha_2\rangle &= \frac{1}{\sqrt{2}}(-1)^{\alpha_1}\alpha_2\left(1-\frac{1}{N-\alpha_1}z\frac{d}{dz}\right)\|z,\alpha_1,\alpha_2-1\rangle \\
&\quad - \frac{1}{\sqrt{2}}(1-\alpha_1)\,(N-\alpha_2)\,z\|z,\alpha_1+1,\alpha_2\rangle
\end{aligned} \tag{4.5}
$$

It is clear that the aforesaid realizations are inhomogeneous. Therefore, it may be of use for quasiexactly solvable problems in quantum mechanics.

We have constructed the simple coherent state of the SPL(2,1) superalgebra and discussed its properties in detail. We also have calculated the matrix elements of the SPL(2,1) generators. The new inhomogeneous differential realizations of the SPL(2,1) generators have been obtained in the coherent-state space.

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